ELSEVIER

Journal of Geometry and Physics 40 (2002) 201–208

JOURNAL OF GEOMETRY AND PHYSICS

www.elsevier.com/locate/jgp

Collapsing of connected sums and the eigenvalues of the Laplacian

Junya Takahashi

Graduate School of Mathematical Sciences, The University of Tokyo, 3-8-1 Komaba, Meguro, Tokyo 153-8914, Japan

Received 26 July 2000; received in revised form 14 March 2001

Abstract

We study the behavior of the eigenvalues of the Laplacian acting on functions when one side of a connected sum of two closed Riemannian manifolds collapses to a point. We prove that the eigenvalues converge to those of the limit space, by using the method of Anné and Colbois. From this, we obtain a gluing theorem for the eigenvalues.

MSC: Primary 58J50; Secondary 35P15, 35P20, 53C23

Keywords: Laplacian; Eigenvalue; Collapsing of Riemannian manifolds

1. Introduction

We have much knowledge of the eigenvalues of the Laplacian acting on functions under collapsings of closed Riemannian manifolds. For a family of Riemannian manifolds with bounded sectional curvature and diameter, Fukaya [11] proved that the eigenvalues converge to those of the limit space with respect to the measured Gromov–Hausdorff topology. Shioya [14] extended it for a family of Alexandrov spaces with curvature bounded below. However, if the curvature is not bounded below, the eigenvalues do not converge in general. We are interested in the cases of the convergence of the eigenvalues for a family of Riemannian manifolds without curvature bounded below. For the collapsings of handles and dumbbells, the convergence of the eigenvalues have been studied by Chavel and Feldman [6,7] and Anné and Colbois [2,4], etc. Colbois and Courtois [8] introduced some topology on a family of pointed Riemannian manifolds and studied the convergence of the eigenvalues with respect to this topology. Their result requires no conditions on curvature.

E-mail address: junya@ms.u-tokyo.ac.jp (J. Takahashi).

PII: S0393-0440(01)00033-X

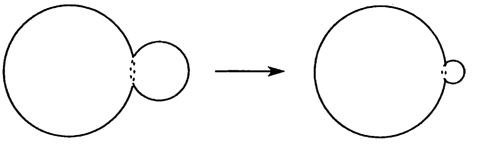

Fig. 1. Collapsing of (M, g_ε).

In the present paper, we study the convergence of the eigenvalues of the Laplacian when one side of a connected sum of two closed Riemannian manifolds collapses to a point, by using the method of Anné and Colbois [1–4] (Fig. 1).

Let (M_i, g_i), $i = 1, 2$, be connected oriented closed Riemannian manifolds of the same dimension m ($m \geq 2$). For simplicity, we suppose that each metric g_i is Euclidean on the geodesic ball $B(x_i, r_i)$ with the radius $r_i > 0$ centered at $x_i \in M_i$, where r_i is smaller than the injectivity radius of (M_i, g_i). Note that this assumption can be omitted by Remark in [4, p. 548]. Furthermore, by changing the scale of g_2, we may suppose $r_2 = 2$. Set $M_i(r) := M_i \setminus B(x_i, r)$. We define the isometry Φ_ε between the boundaries $(\partial M_1(\varepsilon), \partial g_1)$ and $(\partial M_2(1), \varepsilon^2 \partial g_2)$, where ∂g_i is the canonical metric on $\partial M_i(r)$ induced from $(M_i(r), g_i)$. If $S^{m-1}(r)$ is the $(m-1)$-sphere of the radius r in $\mathbf{R}^m$ and h_r its metric, then the identifications $(\partial M_1(\varepsilon), \partial g_1) \cong (S^{m-1}(\varepsilon), h_\varepsilon)$ and $(\partial M_2(1), \varepsilon^2 \partial g_2) \cong (S^{m-1}(1), \varepsilon^2 h_1)$ hold. So we consider Φ_ε as the restriction $(S^{m-1}(\varepsilon), h_\varepsilon) \to (S^{m-1}(1), \varepsilon^2 h_1)$ of the map $\mathbf{R}^m \to \mathbf{R}^m$ with $x \mapsto \varepsilon^{-1}x$. For any ε ($0 < \varepsilon < \min\{r_1, 1\}$), we glue $(M_1(\varepsilon), g_1)$ to $(M_2(1), \varepsilon^2 g_2)$ along Φ_ε. Thus, we obtain the new smooth closed manifold $M := M_1(\varepsilon) \cup_{\Phi_\varepsilon} M_2(1)$ with the piecewise smooth metric

$$g_\varepsilon := \begin{cases} g_1 & \text{on } M_1(\varepsilon), \\ \varepsilon^2 g_2 & \text{on } M_2(1). \end{cases}$$

We choose orientations of M_1 and M_2 such that M is naturally oriented. In [2,4], Anné and Colbois defined the Laplacian on (M, g_ε) and showed that its spectrum consists only of eigenvalue (see Section 2). Since M is connected, the multiplicity of the 0-eigenvalue is 1. Thus, we denote the eigenvalues of the Laplacian on (M, g_ε) by

$$0 = \lambda_0(M, g_\varepsilon) < \lambda_1(M, g_\varepsilon) \leq \cdots \leq \lambda_k(M, g_\varepsilon) \leq \cdots,$$

and similarly for (M_1, g_1). Then, we obtain the following theorem.

Theorem 1.1. *For all $k = 0, 1, \ldots,$ we have*

$$\lim_{\varepsilon \to 0} \lambda_k(M, g_\varepsilon) = \lambda_k(M_1, g_1).$$

For each k, this is uniformly convergent with respect to $j = 0, 1, \ldots, k$. From the proof, we also find the convergence of the associated eigenfunctions. Furthermore, from Theorem 1.1 and the continuity of the eigenvalues with respect to the C^0-topology of metrics, we obtain the following theorem.

Theorem 1.2. *For any $\eta > 0$ and integer $k \geq 0$, there exists a smooth metric $g_{\eta,k}$ on M, which depends on η and k, such that for all $j = 0, 1, \dots, k$,*

$$|\lambda_j(M, g_{\eta,k}) - \lambda_j(M_1, g_1)| < \eta.$$

In the case of dimension $m \geq 3$, we can also obtain Theorem 1.2 by Theorem III.1 in [9]. Note that ours is valid for $m \geq 2$. Our collapsing is different from Fukaya's example in [11, p. 545] and does not converge in the sense of Colbois and Courtois [8]. Finally, the case of the Laplacian acting on differential forms has not been known yet.

The structure of the present paper is as follows. In Section 2, we recall the definitions of the Sobolev spaces and the Laplacian on (M, g_ε). We divide the proof of Theorem 1.1 into two stages. In Section 3, we prove $\limsup_{\varepsilon\to 0} \lambda_k(M, g_\varepsilon) \leq \lambda_k(M_1, g_1)$, and in Section 4, we prove $\lambda_k(M_1, g_1) \leq \liminf_{\varepsilon\to 0} \lambda_k(M, g_\varepsilon)$. In Section 5, we prove Theorem 1.2.

2. Preliminaries

We recall the definitions of the Sobolev spaces and the Laplacian on the C^∞-manifold M with the piecewise smooth metric g_ε, according to [2,4]. The L^2-space on (M, g_ε) is defined as follows.

Definition 2.1.

$$L^2(M, g_\varepsilon) := L^2(M_1(\varepsilon), g_1) \times L^2(M_2(1), \varepsilon^2 g_2).$$

To define the Sobolev spaces H^1 and H^2 on (M, g_ε), we need to impose some gluing conditions on the boundaries of $(M_1(\varepsilon), g_1)$ and $(M_2(1), \varepsilon^2 g_2)$ by means of the trace theorem (see [13, Section 6.4.8]).

Definition 2.2.

$$\begin{aligned} H^1(M, g_\varepsilon) := \{ & f = (f_1, f_2) \in H^1(M_1(\varepsilon), g_1) \times H^1(M_2(1), \varepsilon^2 g_2) | \\ & f_1 \restriction_{\partial M_1(\varepsilon)} = f_2 \restriction_{\partial M_2(1)} \circ \Phi_\varepsilon \quad \text{in } L^2(\partial M_1(\varepsilon), \partial g_1)\}, \\ H^2(M, g_\varepsilon) := \{ & f = (f_1, f_2) \in H^2(M_1(\varepsilon), g_1) \times H^2(M_2(1), \varepsilon^2 g_2) | \\ & f_1 \restriction_{\partial M_1(\varepsilon)} = f_2 \restriction_{\partial M_2(1)} \circ \Phi_\varepsilon \quad \text{in } H^1(\partial M_1(\varepsilon), \partial g_1), \\ & \nu_1(f_1) \restriction_{\partial M_1(\varepsilon)} = -\varepsilon^{-1} \nu_2(f_2) \restriction_{\partial M_2(1)} \circ \Phi_\varepsilon \quad \text{in } L^2(\partial M_1(\varepsilon), \partial g_1)\}. \end{aligned}$$

Here ν_1 and ν_2 are the outward unit vector fields along $(\partial M_1(\varepsilon), \partial g_1)$ and $(\partial M_2(1), \partial g_2)$, respectively. So, $\varepsilon^{-1}\nu_2$ is the unit vector field along $(\partial M_2(1), \varepsilon^2 \partial g_2)$. The inner products on these spaces are defined as the direct sums of those of $(M_1(\varepsilon), g_1)$ and $(M_2(1), \varepsilon^2 g_2)$.

Next, we discuss the Laplacian Δ_ε and the bilinear form Q_ε on (M, g_ε).

Definition 2.3. For $f = (f_1, f_2) \in \mathrm{Dom}(\Delta_\varepsilon) := H^2(M, g_\varepsilon)$, the Laplacian Δ_ε on (M, g_ε) is defined as

$$\Delta_\varepsilon(f_1, f_2) := (\Delta_{g_1} f_1, \Delta_{\varepsilon^2 g_2} f_2),$$

where Δ_g is the Laplacian for the Riemannian metric g.

Definition 2.4. The bilinear forms q_{g_1} and q_{g_2} are defined as

$$q_{g_1}(f_1, h_1) := \int_{M_1(\varepsilon)} \langle \mathrm{d}f_1, \mathrm{d}h_1 \rangle_{g_1} \,\mathrm{d}\mu_{g_1} \quad \text{for } f_1, h_1 \in H^1(M_1(\varepsilon), g_1),$$

$$q_{g_2}(f_2, h_2) := \int_{M_2(1)} \langle \mathrm{d}f_2, \mathrm{d}h_2 \rangle_{g_2} \,\mathrm{d}\mu_{g_2} \quad \text{for } f_2, h_2 \in H^1(M_2(1), g_2).$$

The bilinear form Q_ε on $\mathrm{Dom}(Q_\varepsilon) := H^1(M, g_\varepsilon)$ is defined as

$$\begin{aligned} Q_\varepsilon(f, h) &:= q_{g_1}(f_1, h_1) + q_{\varepsilon^2 g_2}(f_2, h_2) \\ &= \int_{M_1(\varepsilon)} \langle \mathrm{d}f_1, \mathrm{d}h_1 \rangle_{g_1} \,\mathrm{d}\mu_{g_1} + \varepsilon^{m-2} \int_{M_2(1)} \langle \mathrm{d}f_2, \mathrm{d}h_2 \rangle_{g_2} \,\mathrm{d}\mu_{g_2} \end{aligned}$$

for $f = (f_1, f_2), h = (h_1, h_2) \in \mathrm{Dom}(Q_\varepsilon)$.

Lemma 2.5. *The bilinear form Q_ε is induced from the Laplacian Δ_ε, i.e. for $f = (f_1, f_2) \in \mathrm{Dom}(\Delta_\varepsilon)$ and $h = (h_1, h_2) \in \mathrm{Dom}(Q_\varepsilon)$,*

$$Q_\varepsilon(f, h) = (\Delta_\varepsilon f, h)_{L^2(M, g_\varepsilon)}.$$

Proof. From the definition of Δ_ε and the Green formula (cf. [15, p. 158]), it follows that

$$\begin{aligned} (\Delta_\varepsilon f, h)_{L^2(M, g_\varepsilon)} &= (\Delta_{g_1} f_1, h_1)_{L^2(M_1(\varepsilon), g_1)} + \varepsilon^{m-2} (\Delta_{g_2} f_2, h_2)_{L^2(M_2(1), g_2)} \\ &= (\mathrm{d}f_1, \mathrm{d}h_1)_{L^2(M_1(\varepsilon), g_1)} + \varepsilon^{m-2} (\mathrm{d}f_2, \mathrm{d}h_2)_{L^2(M_2(1), g_2)} \\ &\quad - \int_{\partial M_1(\varepsilon)} \nu_1(f_1) h_1 \,\mathrm{d}\mu_{\partial g_1} - \varepsilon^{m-2} \int_{\partial M_2(1)} \nu_2(f_2) h_2 \,\mathrm{d}\mu_{\partial g_2}. \end{aligned}$$

By the gluing conditions $\nu_2(f_2) \circ \Phi_\varepsilon = -\varepsilon \nu_1(f_1)$, $h_2 \circ \Phi_\varepsilon = h_1$ and $\Phi_\varepsilon^*(\mathrm{d}\mu_{\partial g_2}) = \varepsilon^{-m+1} \,\mathrm{d}\mu_{\partial g_1}$, we obtain

$$\begin{aligned} \varepsilon^{m-2} \int_{\partial M_2(1)} \nu_2(f_2) h_2 \,\mathrm{d}\mu_{\partial g_2} &= \varepsilon^{m-2} \int_{\partial M_1(\varepsilon)} \nu_2(f_2) \circ \Phi_\varepsilon \cdot h_2 \circ \Phi_\varepsilon \cdot \Phi_\varepsilon^*(\mathrm{d}\mu_{\partial g_2}) \\ &= - \int_{\partial M_1(\varepsilon)} \nu_1(f_1) h_1 \,\mathrm{d}\mu_{\partial g_1}. \end{aligned}$$

Hence, we have finished the proof. □

By Section 1 in [4], Δ_ε has the properties of the Laplacian on smooth closed Riemannian manifolds. Namely, Δ_ε is a non-negative, self-adjoint elliptic operator and its spectrum consists only of the eigenvalues with finite multiplicity. We denote by $\lambda_k(M, g_\varepsilon)$ or $\lambda_k(Q_\varepsilon)$ the kth eigenvalue of the Laplacian Δ_ε.

3. Proof of Theorem 1.1, Part I

In this section, we prove $\limsup_{\varepsilon\to 0} \lambda_k(Q_\varepsilon) \le \lambda_k(M_1, g_1)$, by using the min–max principle. Let f_i be the ith eigenfunction on (M_1, g_1) with the eigenvalue $\lambda_i(M_1, g_1)$, such that all f_i are orthonormal $(i = 0, 1, \dots, k)$. We define a cut-off function $\chi_\varepsilon : [0, \infty) \to [0, 1]$ as

$$\chi_\varepsilon(r) := \begin{cases} 0 & (0 \le r \le \varepsilon), \\ -\dfrac{2}{\log \varepsilon} \log\left(\dfrac{r}{\varepsilon}\right) & (\varepsilon \le r \le \sqrt{\varepsilon}), \\ 1 & (\sqrt{\varepsilon} \le r), \end{cases}$$

which was introduced in [10], Proposition 1.3.1 (see also [3, Section 6]). We set $\chi_\varepsilon(x) := \chi_\varepsilon(d_{g_1}(x_1, x))$ for $x \in M_1$, where d_{g_1} is the distance induced from g_1.

Let E_ε be the subspace spanned by $\{\chi_\varepsilon f_0, \dots, \chi_\varepsilon f_k\}$ in $H_0^1(M_1(\varepsilon)^\circ, g_1)$. Then, we can consider E_ε as the subspace of $H^1(M, g_\varepsilon)$ by the 0-extension $f \mapsto (f, 0)$. Hence, by the min–max principle for Q_ε, we have the inequality

$$\lambda_k(Q_\varepsilon) \le \sup_{u_\varepsilon \ne 0 \in E_\varepsilon} \left\{ \frac{q_{g_1}(u_\varepsilon, u_\varepsilon)}{\|u_\varepsilon\|^2_{L^2(M_1(\varepsilon), g_1)}} \right\}.$$

Since $m \ge 2$, we have

$$\int_{B(x_1, \sqrt{\varepsilon})} |\mathrm{d}\chi_\varepsilon|^2_{g_1} \, \mathrm{d}\mu_{g_1} = \frac{4 \operatorname{vol}(S^{m-1}(1))}{(\log \varepsilon)^2} \int_\varepsilon^{\sqrt{\varepsilon}} r^{m-3} \, \mathrm{d}r \to 0,$$

as $\varepsilon \to 0$. Hence, in the same way as (4.2) in [3], we obtain

$$\lambda_k(Q_\varepsilon) \le \lambda_k(M_1, g_1) + \delta(\varepsilon),$$

where $\delta(\varepsilon) \to 0$ as $\varepsilon \to 0$, i.e., we see that $\limsup_{\varepsilon\to 0} \lambda_k(Q_\varepsilon) \le \lambda_k(M_1, g_1)$.

4. Proof of Theorem 1.1, Part II

In this section, we prove $\lambda_k(M_1, g_1) \le \liminf_{\varepsilon\to 0} \lambda_k(Q_\varepsilon)$. Throughout this section, we denote by C a generic positive constant independent of the functions and the indices.

Let $f_{j,\varepsilon} = (f^1_{j,\varepsilon}, f^2_{j,\varepsilon}) \in \mathrm{Dom}(Q_\varepsilon)$ $(j = 0, 1, \dots, k)$ be orthonormal eigenfunctions with the eigenvalue $\lambda_j(Q_\varepsilon)$, i.e.,

$$Q_\varepsilon(f_{j,\varepsilon}, f_{j,\varepsilon}) = \lambda_j(Q_\varepsilon) \|f_{j,\varepsilon}\|^2_{L^2(M, g_\varepsilon)}, \qquad (f_{i,\varepsilon}, f_{j,\varepsilon})_{L^2(M, g_\varepsilon)} = \delta_{ij}.$$

We set $\alpha_j := \liminf_{\varepsilon\to 0} \lambda_j(Q_\varepsilon)$. Our purpose is to show $\lambda_k(M_1, g_1) \le \alpha_k$.

By using the harmonic extension of (4.3) in [3] (see also [12, p. 40]), we have an extension $\bar{f}^1_{j,\varepsilon} \in H^1(M_1, g_1)$ of $f^1_{j,\varepsilon}$, such that

$$\|\bar{f}^1_{j,\varepsilon}\|_{H^1(M_1, g_1)} \le C \|f^1_{j,\varepsilon}\|_{H^1(M_1(\varepsilon), g_1)}.$$

The family $\{\bar{f}^1_{j,\varepsilon}\}_{\varepsilon>0}$ is bounded in $H^1(M_1, g_1)$, since

$$\begin{aligned}\|\bar{f}^1_{j,\varepsilon}\|^2_{H^1(M_1,g_1)} &\le C\|f^1_{j,\varepsilon}\|^2_{H^1(M_1(\varepsilon),g_1)} = C\{\|f^1_{j,\varepsilon}\|^2_{L^2(M_1(\varepsilon),g_1)} + q_{g_1}(f^1_{j,\varepsilon}, f^1_{j,\varepsilon})\}\\ &\le C\{1 + Q_\varepsilon(f_{j,\varepsilon}, f_{j,\varepsilon})\} \le C\{1 + \lambda_j(M_1, g_1)\},\end{aligned} \tag{4.1}$$

where the last inequality follows from the result of Section 3. Hence, by the weakly convergence theorem, there exist a subsequence $\{\bar{f}^1_{j,\varepsilon_i}\}_{i=1}^\infty$ and $\bar{f}^1_j$ in $H^1(M_1, g_1)$, such that $\lambda_j(Q_{\varepsilon_i}) \to \alpha_j$ and $\bar{f}^1_{j,\varepsilon_i} \to \bar{f}^1_j$ weakly in $H^1(M_1, g_1)$, as $i \to \infty$, i.e., $\varepsilon_i \to 0$. Furthermore, since the embedding $H^1(M_1, g_1) \subset L^2(M_1, g_1)$ is compact by the Rellich theorem, $\{\bar{f}^1_{j,\varepsilon_i}\}_i$ also converges strongly in $L^2(M_1, g_1)$.

For any $\varphi \in C_0^\infty(M_1 \setminus \{x_1\})$, the space of smooth functions with compact support, we obtain

$$\begin{aligned}q_{g_1}(\bar{f}^1_j, \varphi) &= \lim_{i\to\infty}\{q_{g_1}(f^1_{j,\varepsilon_i}, \varphi) + q_{\varepsilon_i^2 g_2}(f^2_{j,\varepsilon_i}, 0)\} = \lim_{i\to\infty} Q_{\varepsilon_i}((f^1_{j,\varepsilon_i}, f^2_{j,\varepsilon_i}), (\varphi, 0))\\ &= \lim_{i\to\infty} \lambda_j(Q_{\varepsilon_i})((f^1_{j,\varepsilon_i}, f^2_{j,\varepsilon_i}), (\varphi, 0))_{L^2(M,g_{\varepsilon_i})}\\ &\qquad \text{(since } f_{j,\varepsilon_i} \text{ is the eigenfunction of } Q_{\varepsilon_i})\\ &= \alpha_j(\bar{f}^1_j, \varphi)_{L^2(M_1,g_1)}.\end{aligned}$$

Since $C_0^\infty(M_1 \setminus \{x_1\})$ is dense in $H^1(M_1, g_1)$ (see [1]), we have $q_{g_1}(\bar{f}^1_j, \varphi) = \alpha_j (\bar{f}^1_j, \varphi)_{L^2(M_1,g_1)}$ for any $\varphi \in H^1(M_1, g_1)$. From the regularity theorem for weak solutions, it follows that $\bar{f}^1_j \in C^\infty(M_1)$ and $\Delta_{g_1}\bar{f}^1_j = \alpha_j \bar{f}^1_j$. If $\{\bar{f}^1_0, \dots, \bar{f}^1_k\}$ are orthonormal, then α_k is the lth eigenvalue of (M_1, g_1) for some $l \ge k$. Thus, we have $\lambda_k(M_1, g_1) \le \alpha_k$.

Now, we prove that $\{\bar{f}^1_0, \dots, \bar{f}^1_k\}$ are orthonormal. Set $\tilde{f}^2_{j,\varepsilon} := \varepsilon^{m/2} f^2_{j,\varepsilon}$. In the same way as (4.1), $\{\tilde{f}^2_{j,\varepsilon_i}\}_i$ is bounded in $H^1(M_2(1), g_2)$, and hence there exist a subsequence of $\{\tilde{f}^2_{j,\varepsilon_i}\}_i$ and $\tilde{f}^2_j$ in $H^1(M_2(1), g_2)$ such that $\tilde{f}^2_{j,\varepsilon_i} \to \tilde{f}^2_j$ weakly in $H^1(M_2(1), g_2)$ and strongly in $L^2(M_2(1), g_2)$ as $i \to \infty$.

Claim 4.1. For $j = 0, \dots, k$, we have $\tilde{f}^2_j = 0$.

If this claim holds, we see that $\{\bar{f}^1_0, \dots, \bar{f}^1_k\}$ are orthonormal. In fact,

$$\begin{aligned}(\bar{f}^1_j, \bar{f}^1_l)_{L^2(M_1,g_1)} &= \lim_{i\to\infty}\{(f^1_{j,\varepsilon_i}, f^1_{l,\varepsilon_i})_{L^2(M_1(\varepsilon_i),g_1)} + (\bar{f}^1_{j,\varepsilon_i}, \bar{f}^1_{l,\varepsilon_i})_{L^2(B(x_1,\varepsilon_i),g_1)}\}\\ &= \lim_{i\to\infty}\{(f_{j,\varepsilon_i}, f_{l,\varepsilon_i})_{L^2(M,g_{\varepsilon_i})} - (f^2_{j,\varepsilon_i}, f^2_{l,\varepsilon_i})_{L^2(M_2(1),\varepsilon_i^2 g_2)}\}\\ &= \delta_{jl} - \lim_{i\to\infty}(\tilde{f}^2_{j,\varepsilon_i}, \tilde{f}^2_{l,\varepsilon_i})_{L^2(M_2(1),g_2)} = \delta_{jl},\end{aligned}$$

where the last equality follows from Claim 4.1.

Now, we prove Claim 4.1. From $\tilde{f}^2_{j,\varepsilon_i} \to \tilde{f}^2_j$ weakly in $H^1(M_2(1), g_2)$ and strongly in $L^2(M_2(1), g_2)$ as $i \to \infty$, it follows that:

$$\begin{aligned}\|\mathrm{d}\tilde{f}^2_j\|^2_{L^2(M_2(1),g_2)} &\le \liminf_{i\to\infty}\|\mathrm{d}\tilde{f}^2_{j,\varepsilon_i}\|^2_{L^2(M_2(1),g_2)} \le \liminf_{i\to\infty} \varepsilon_i^2 Q_{\varepsilon_i}(f_{j,\varepsilon_i}, f_{j,\varepsilon_i})\\ &= \liminf_{i\to\infty} \varepsilon_i^2 \lambda_j(Q_{\varepsilon_i}) = 0,\end{aligned}$$

i.e., $\tilde{f}_j^2$ is a constant. Furthermore, we see that

$$\|\tilde{f}_{j,\varepsilon_i}^2 - \tilde{f}_j^2\|_{H^1(M_2(1),g_2)}^2 = \|\tilde{f}_{j,\varepsilon_i}^2 - \tilde{f}_j^2\|_{L^2(M_2(1),g_2)}^2 + \|\mathrm{d}\tilde{f}_{j,\varepsilon_i}^2\|_{L^2(M_2(1),g_2)}^2 \to 0 \ \ (\text{as}\, i \to \infty). \tag{4.2}$$

On the other hand, we see $\tilde{f}_j^2 \restriction_{\partial M_2(1)} = 0$. In fact, from the gluing condition of f_{j,ε_i} (see Definition 2.2), we have

$$\begin{aligned}\|\tilde{f}_{j,\varepsilon_i}^2 \restriction_{\partial M_2(1)}\|_{L^2(\partial M_2(1),\partial g_2)} &= \sqrt{\varepsilon_i}\|f_{j,\varepsilon_i}^2 \restriction_{\partial M_2(1)}\|_{L^2(\partial M_2(1),\varepsilon_i^2\partial g_2)} \\ &= \sqrt{\varepsilon_i}\|f_{j,\varepsilon_i}^1 \restriction_{\partial M_1(\varepsilon_i)}\|_{L^2(\partial M_1(\varepsilon_i),\partial g_1)} \\ &\le \begin{cases} C\varepsilon_i\sqrt{|\log\varepsilon_i|}\|f_{j,\varepsilon_i}^1\|_{H^1(M_1(\varepsilon_i),g_1)} & \text{if } m = 2, \\ C\varepsilon_i\|f_{j,\varepsilon_i}^1\|_{H^1(M_1(\varepsilon_i),g_1)} & \text{if } m \ge 3, \end{cases}\end{aligned}$$

where the last inequalities are obtained by Anné [2]. Since $\|f_{j,\varepsilon_i}^1\|_{H^1(M_1(\varepsilon_i),g_1)}$ is bounded by (4.1), we have $\|\tilde{f}_{j,\varepsilon_i}^2 \restriction_{\partial M_2(1)}\|_{L^2(\partial M_2(1),\partial g_2)} \to 0$ as $i \to \infty$. Hence, from the trace theorem and (4.2), it follows that:

$$\begin{aligned}&\|\tilde{f}_j^2 \restriction_{\partial M_2(1)}\|_{L^2(\partial M_2(1),\partial g_2)} \\ &\quad \le \|\tilde{f}_{j,\varepsilon_i}^2 \restriction_{\partial M_2(1)}\|_{L^2(\partial M_2(1),\partial g_2)} + \|\tilde{f}_j^2 \restriction_{\partial M_2(1)} - \tilde{f}_{j,\varepsilon_i}^2 \restriction_{\partial M_2(1)}\|_{L^2(\partial M_2(1),\partial g_2)} \\ &\quad \le \|\tilde{f}_{j,\varepsilon_i}^2 \restriction_{\partial M_2(1)}\|_{L^2(\partial M_2(1),\partial g_2)} + C\|\tilde{f}_j^2 - \tilde{f}_{j,\varepsilon_i}^2\|_{H^1(M_2(1),g_2)} \to 0,\end{aligned}$$

as $i \to \infty$, i.e., $\tilde{f}_j^2 \restriction_{\partial M_2(1)} = 0$. Since $\tilde{f}_j^2$ is a constant, we obtain $\tilde{f}_j^2 = 0$.

Therefore, we see that $\{\bar{f}_0^1, \dots, \bar{f}_k^1\}$ is orthonormal, hence, we have finished the proof that $\lambda_k(M_1, g_1) \le \alpha_k$.

5. Proof of Theorem 1.2

In this section, we prove Theorem 1.2. We use notation as in Section 1. From Theorem 1.1, for any $\eta > 0$ and $k \ge 0$, there is an $\varepsilon_0 > 0$, such that for the associated piecewise smooth metric g_{ε_0} on M and $j = 0, 1, \dots, k$,

$$|\lambda_j(M, g_{\varepsilon_0}) - \lambda_j(M_1, g_1)| < \tfrac{1}{2}\eta. \tag{5.1}$$

On the other hand, there exists a sequence of smooth metrics $\{g_i\}_{i=1}^\infty$ on M such that $g_i \to g_{\varepsilon_0}$ with respect to the C^0-topology as $i \to \infty$. For the piecewise smooth metrics such as g_{ε_0}, the eigenvalues of the Laplacian also depend continuously on metrics with respect to the C^0-topology (see [5, p. 162]). Hence, there exists an $i_0 > 0$ such that for the smooth metric g_{i_0}

$$|\lambda_j(M, g_{i_0}) - \lambda_j(M, g_{\varepsilon_0})| < \tfrac{1}{2}\eta. \tag{5.2}$$

Therefore, if we set the smooth metric $g_{\eta,k} := g_{i_0}$, from (5.1) and (5.2), we obtain

$$|\lambda_j(M, g_{\eta,k}) - \lambda_j(M_1, g_1)| < \eta,$$

for $j = 0, 1, \dots, k$.

Acknowledgements

The author is grateful to Prof. Takushiro Ochiai and Prof. Hiroshi Konno for their valuable comments and constant encouragement. The author thanks Prof. Colette Anné for the valuable letter, Prof. Kazuhiro Kuwae and Prof. Takashi Shioya for their valuable comments, and Dr. Katsuhiro Yoshiji for valuable discussions. Finally, the author thanks the referee who suggested the improvement of the proof of Claim 4.1.

References

[1] C. Anné, Perturbation du $X - TUB^{\varepsilon} Y$ (conditions de Neumann), Séminarie de Théorie Spectre et Géométrie, Année 1985–1986, Vol. 4, Univ. Grenoble, 1986, pp. 17–23.
[2] C. Anné, Spectre du Laplacien et écracement d'anses, Ann. Sci. Éc. Norm. Sup. 20 (4) (1987) 271–280.
[3] C. Anné, B. Colbois, Opérateur de Hodge–Laplace sur des variétés compactes privées d'un nombre fini de boules, J. Func. Anal. 115 (1993) 143–160.
[4] C. Anné, B. Colbois, Spectre du Laplacien agissant sur les p-formes différentielles et écrasement d'anses, Math. Ann. 303 (1995) 545–573.
[5] S. Bando, H. Urakawa, Generic properties of the eigenvalues of the Laplacian for compact Riemannian manifolds, Tôhoku Math. J. 35 (1983) 155–172.
[6] I. Chavel, E. Feldman, Spectra of manifolds with small handles, Comment. Math. Helv. 56 (1981) 83–102.
[7] I. Chavel, Eigenvalues in Riemannian Geometry, Vol. 115, Academic Press, New York, 1984.
[8] B. Colbois, G. Courtois, Convergence de variétés et convergence du spectre du Laplacien, Ann. Sci. Éc. Norm. Sup. 24 (4) (1991) 507–518.
[9] Y. Colin de Verdière, Sur la multiplcité de la premère valeur propre non nulle du Laplacien, Comm. Math. Helv. 61 (1986) 254–270.
[10] G. Courtois, Comportement du spectre d'une variété Riemannienne compacte sous perturbation topologique par excision d'un domaine, Thèse, Institut Fourier, Grenoble, 1987.
[11] K. Fukaya, Collapsing of Riemannian manifolds and eigenvalues of Laplace operator, Invent. Math. 87 (1987) 517–547.
[12] J. Rauch, M. Taylor, Potential and scattering theory on wildly perturbed domains, J. Func. Anal. 18 (1975) 27–59.
[13] M. Renardy, R.C. Rogers, An Introduction to Partial Differential Equations: Texts in Applied Mathematics, Vol. 13, Springer, Berlin, 1992.
[14] T. Shioya, Convergence of Alexandrov spaces and spectrum of Laplacian, J. Math. Soc. Jpn. 53 (2001) 1–15.
[15] F.W. Warner, Foundations of Differentiable Manifolds and Lie Groups, GTM 94, Springer, Berlin, 1983.